WEIGHTED HARDY INEQUALITY ON RIEMANNIAN MANIFOLDS

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Abstract. Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$ and we let Σ to be a closed submanifold of dimension $1 \leq k \leq N-2$. In this paper we study existence and non-existence of minimizers of Hardy inequality with weight function singular on Σ within the framework of Brezis-Marcus-Shafrir [8]. In particular we provide necessary and sufficient conditions for existence of minimizers.

1 Introduction

Let $N \geq 3$, $1 \leq k \leq N-2$ and pose $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. We denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$u \longmapsto \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

We recall the following Hardy type inequality with cylindrical weights

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-k-2}{2}\right)^2 \int_{\mathbb{R}^N} |y|^{-2}|u|^2 dx, \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.1}$$

see the book of May'ja [48] for a proof. See also the work of Brezis-Vasquez [9], Musina [39] and Gazzini-Musina [33]. The constant $\left(\frac{N-k-2}{2}\right)^2$ is sharp and never achieved in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. However there exists a function

$$u(x) = |y|^{\frac{2+k-N}{2}}$$

which satisfies

$$\Delta_{\mathbb{R}^N} u + \left(\frac{N-k-2}{2}\right)^2 |y|^{-2} u = 0. \tag{1.2}$$

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We will call this function a ”virtual” ground state because it does not belong to $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Inequality (1.1) is invariant by translation in the z-variable and by scaling in the full variable yielding difficulties in the study of elliptic and parabolic equations involving inverse square potentials.
Note that (1.1) is not in general valid for a Riemannian manifold $(\mathcal{M}^N, g)$ of dimension $N \geq 3$. However, by Allegretto-Piepenbrink argument (see [3] and [40]) and by construction of supersolution near Σ, we prove the local Hardy inequality below in Lemma 3.1, in a small tubular neighborhood

$$\Sigma_r := \{p \in \mathcal{M} : \rho(p) := dist(p, \Sigma) < r\}$$

of Σ, i.e

$$\int_{\Sigma_r} |\nabla u| dv_g \geq \left(\frac{N-k-2}{2}\right)^2 \int_{\Sigma_r} \rho^{-2}|u|^2 dv_g, \ \ \forall u \in H^1(\Sigma_r), \tag{1.3}$$

where $\rho(p) := dist(p, \Sigma)$ is the geodesic distance to Σ. This type of result was first proved by Brezis-Marcus in [7]. See also the work of Fall-Mahmoudi in [20] and Thiam in [53].
Using (1.3) with an argument of partition of unity around Σ, we will prove in Lemma 3.2 below the following

$$\int_{\mathcal{M}} |\nabla u|_g^2 dv_g \geq \left(\frac{N-k-2}{2}\right)^2 \int_{\mathcal{M}} \rho^{-2}|u|^2 dv_g + \lambda \int_{\mathcal{M}} u^2 dv_g, \ \ \forall u \in H^1(\mathcal{M}) \tag{1.4}$$

for a constant λ depending on $\mathcal{M}$. We remark that the Hardy inequality is a particular case of the Caffarelli-Kohn-Nirenberg inequality, see [10]. The knowledge of Hardy, Hardy-Sobolev, Gagliardo-Nirenberg, Sobolev or Caffarelli-Kohn-Nirenberg inequality on a manifold $\mathcal{M}$ and their best constants allows to obtain qualitative properties on the manifold $\mathcal{M}$. For instance in [2], [11] and [54] it was shown that if $\mathcal{M}$ is a complete open Riemannian manifold with non negative Ricci curvature in which a Hardy or Gagliardo-Nirenberg or Caffarelli-Kohn-Nirenberg type inequality holds, then $\mathcal{M}$ is in some suitable sense close to the Euclidean space.
Inequalities involving integrals of a function and its derivatives together with singular weights appear frequently in various branches of mathematics and represent a useful tool in the theory of differential equations. They have several applications in many questions from mathematical physics, spectral theory, analysis of linear and nonlinear PDEs, harmonic analysis and stochastic analysis. For more details related to these inequalities, in particular the Hardy one, see [2, 4, 6, 7, 12, 13, 14, 16, 18, 19, 20, 22, 23, 30, 34, 35, 38, 45, 47].
In this paper, we are interested in the following Hardy inequality with weight functions on a compact Riemannian manifold $(\mathcal{M}, g)$ of dimension N. Therefore we propose to study the problem of finding minimizers of the following quotient in the spirit of Brezis-Marcus [7]

$$\mu_\lambda(\mathcal{M}, \Sigma, b, q, \eta) := \inf_{u \in H^1(M)} \frac{\displaystyle\int_{\mathcal{M}} b|\nabla u|_g^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2}|u|^2 \eta dv_g}{\displaystyle\int_{\mathcal{M}} \rho^{-2}|u|^2 q dv_g}, \tag{1.5}$$

where $\rho(p) := \mathrm{dist}(p, \Sigma)$ is the geodesic distance function to Σ and the weights functions $b,\ q$ and η satisfy

$$b, q \in \mathcal{C}^2(\mathcal{M}), \quad b, q > 0 \quad \text{in} \quad \mathcal{M}, \quad \eta > 0 \quad \text{in} \quad \mathcal{M} \setminus \Sigma, \quad \eta \in Lip(\mathcal{M}) \tag{1.6}$$

and

$$\max_{\Sigma} \frac{q}{b} = 1, \qquad \eta = 0 \quad \text{on} \quad \Sigma. \tag{1.7}$$

We have the following

Theorem 1.1 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$ and let $\Sigma \subset \mathcal{M}$ be a closed submanifold of dimension $k = 1, ..., N-2$. Assume that the weight functions b, q and η satisfy* (1.6) *and* (1.7)*. Then, there exists $\lambda^* = \lambda^*(b, q, \eta, \mathcal{M}, \Sigma)$ such that*

$$\mu_\lambda(\mathcal{M}, \Sigma) = \left(\frac{N-k-2}{2} \right)^2, \quad \forall \lambda \leq \lambda^*, \tag{1.8}$$

$$\mu_\lambda(\mathcal{M}, \Sigma) < \left(\frac{N-k-2}{2} \right)^2, \quad \forall \lambda > \lambda^*. \tag{1.9}$$

The infimum $\mu_\lambda(\mathcal{M}, \Sigma)$ is attained if $\lambda > \lambda^*$ and it is not attained when $\lambda < \lambda^*$.

The existence of λ^* is a consequence of the local Hardy inequality

$$\int_{\Sigma_r} b|\nabla u|^2 dv_g \geq \left(\frac{N-k-2}{2} \right)^2 \int_{\Sigma_r} q|u|^2 \rho^{-2} dv_g \tag{1.10}$$

(see Lemma 3.1 and Lemma 3.2). The existence and non-existence parts are classic. They were almost the same done in [7] and in [53]. A natural question is to know what happens concerning the critical case. Thus, we have the following

Theorem 1.2 *let λ^* be given by Theorem 1.1. Then $\mu_{\lambda^*}(\mathcal{M}, \Sigma)$ is achieved if and only if*

$$\int_{\Sigma} \frac{d\sigma}{\sqrt{1 - q(\sigma)/b(\sigma)}} < \infty. \tag{1.11}$$

As a consequence of this, we get the following

Corollary 1.3 *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension $N \geq 3$ and let Σ be a closed submanifold of dimension $1 \leq k \leq N-2$. For $\lambda \in \mathbb{R}$, put*

$$\mathcal{V}_\lambda(\mathcal{M}, \Sigma) := \inf_{u \in H^1(\mathcal{M})} \frac{\displaystyle\int_{\mathcal{M}} |\nabla u|^2 dv_g - \lambda \int_{\mathcal{M}} u^2 dv_g}{\displaystyle\int_{\mathcal{M}} \rho^{-2} u^2 dv_g}. \tag{1.12}$$

Then there exists $\lambda^* = \lambda^*(\mathcal{M}, \Sigma)$ such that

$$\mathcal{V}_\lambda(\mathcal{M}, \Sigma) = \left(\frac{N-k-2}{2}\right)^2, \ \ \forall \lambda \leq \lambda^*,$$

$$\mathcal{V}_\lambda(\mathcal{M}, \Sigma) < \left(\frac{N-k-2}{2}\right)^2, \ \ \forall \lambda > \lambda^*.$$

Moreover $\mathcal{V}_\lambda(\mathcal{M}, \Sigma)$ is attained if and only if $\lambda > \lambda^*$.

When the singularity is reduced to a single point $\{p_0\}$ ($k = 0$), the corollary remain valid. It was proved by Thiam in [53].
Our arguments of proof are based on the construction of a H^1 super-solution and a H^1 sub-solution of the linear operator L_λ defined in (2.34). Without any loss of generality we may assume that $b \equiv 1$ (see Section 4 below) and r is small enough and we perturb the virtual ground-state

$$v_{a,q}(p) = (-log\rho)^a \rho^\alpha(p)$$

for the Hardy constant $\left(\frac{N-k-2}{2}\right)^2$, where

$$\alpha(x) = \frac{2+k-N}{2}\left(1 - \sqrt{1 - q(\sigma(x)) + |x|}\right).$$

Furthermore, it's easy to verify that for $a < -\frac{1}{2}$ and for $\varepsilon \in (0,1)$, $v_{a,q}$ and $v_{0,q-\varepsilon}$ belong to $H^1(\Sigma_r)$. We prove the non-existence part by assuming by contradiction that, when

$$\int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}} = \infty, \tag{1.13}$$

there exists a non-negative solution $u \in H^1(\mathcal{M}) \cap \mathcal{C}(\mathcal{M} \setminus \Sigma)$. We then construct a $H^1(\Sigma_r)$ sub-solution $V_\varepsilon := v_{-1,q} + v_{0,q-\varepsilon}$ which is upper bounded by u (modulo a multiplicative positive constant independent on a and ε) so that

$$||\rho^{-1} V_\varepsilon||_{L^2(\Sigma_r)} \leq C||\rho^{-1} u||_{L^2(\mathcal{M})} \leq C'||u||_{H^1(\mathcal{M})} \tag{1.14}$$

by the Hardy inequality (1.4). Moreover using polar coordinates we verify that

$$C \int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}} \leq \int_{\Sigma_r} V_0^2 \rho^{-2} dv_g \tag{1.15}$$

for r small enough. Hence taking the limit in (1.14) as $\varepsilon \longrightarrow 0$, we get contradiction.
For the existence part, we construct a super-solution $U := v_{0,q} - v_{-1,q}$ and we suppose that

$$\int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}} < +\infty. \tag{1.16}$$

Then $U \in H^1(\Sigma_r)$, (see Lemma 2.3 below). Next, we let the sequence of real numbers $\{\lambda_n\}$ decreasing to λ^*. By Theorem 1.1, we can now associate to each λ_n a positive minimizer $u_n \in H^1(\mathcal{M}) \cap \mathcal{C}(\mathcal{M} \setminus \Sigma)$ for μ_{λ_n}. Then using some comparison argument, the sequence $\{u_n\}$ is uniformly bounded in Σ_{r_0} by the super-solution U (modulo a multiplicative positive constant independent on n). Hence $\rho^{-1}u_n$ converge strongly to $\rho^{-1}u$ in $L^2(\mathcal{M})$ by Rellich-Kondrakov theorem and that u_n converge to u in $H^1(\mathcal{M})$ strongly.
The paper is organized as follows. In Section 2, we give some Preliminaries and Notations and we construct a super and a sub-solutions we will use in Section 4 to prove Theorem 1.2 and in Section 3, we prove the existence of λ^* and we give a complete proof of Theorem 1.1.

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2 Preliminaries and Notations

Consider $p \in \Sigma$. We denote by $T_p\Sigma$ the tangent space of Σ and $N_p\Sigma$ he normal space of $T_p\Sigma$ at p. We may assume that

$$N_p\Sigma = \text{Span}\big\langle E_1, ..., E_{N-k}\big\rangle \quad \text{and} \quad T_p\Sigma = \text{Span}\big\langle E_{N-k+1}, ..., E_N\big\rangle. \tag{2.1}$$

A neighborhood of p in Σ can be parametrized via the mapping

$$\text{Exp}_p^{\Sigma} : B_r(0) \subset \mathbb{R}^k \to \Sigma_r \supset \Sigma \tag{2.2}$$

$$y \longmapsto f^p(z) = \text{Exp}_p^{\Sigma}\bigg(\textstyle\sum_{a=N-k+1}^{N} z_a E_a\bigg),$$

where $z = (z_{N-k+1}, ..., z_N) \in \mathbb{R}^k$, $B_r(0)$ is the ball centered at 0 and of radius r, Exp_p^{Σ} is the exponential mapping at p in Σ and Σ_r defined in (2.4). Now we extend $(E_i)_{1\le i\le N-k}$ to an orthonormal frame $(X_i)_{1\le i\le N-k}$ in a neighborhood of p in $\mathcal{M}$ via the mapping

$$\text{Exp}_{f_p(z)}^{\mathcal{M}} : \mathbb{R}^k \times \mathbb{R}^{N-k} \to \mathcal{M} \tag{2.3}$$

$$x = (y, z) \longmapsto F_{\mathcal{M}}^{p_i}(x) = \text{Exp}_{f_p(z)}^{\mathcal{M}}\bigg(\textstyle\sum_{i=1}^{N-k} y_i X_i\bigg),$$

where $y = (y_1, ..., y_{N-k})$ and $\text{Exp}_{f_p(z)}^{\mathcal{M}}$ is the exponential map at $f_p(z)$ in $\mathcal{M}$.
In the following, we will consider the geodesic neighborhood contained in $\mathcal{M}$ around Σ of radius r

$$\Sigma_r = \big\{p \in \mathcal{M} : \rho(p) := dist(p, \Sigma) < r\big\}. \tag{2.4}$$

In these normal coordinates, the Laplace-Beltrami operator is given by

$$\Delta_g = -g^{ij}\left(\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} - \Gamma_{ij}^k \frac{\partial}{\partial_{x_k}}\right), \tag{2.5}$$

where $\{\Gamma_{ij}^k\}_{1\le i,j,k\le N}$ are the components of the metric g and $g^{ij} = (-g^{-1})_{ij}$ are the components of the inverse matrix of g. Then the following estimates hold

$$\Gamma_{ij}^k(x) = O(|y|), \qquad g_{ij}(F_{\mathcal{M}}^{p_i}(x)) = \delta_{ij} + O(|y|^2) \quad \text{and} \qquad \rho_k(F_{\mathcal{M}}^{p_i}(x)) = |y|, \tag{2.6}$$

see the paper of Mahmoudi-Mazzeo-Pacard [44]. In addition, there exists a positive constant r_0 depending on Σ and $\mathcal{M}$ such that $\rho \in \mathcal{C}_c^\infty(\Sigma_r)$. Moreover Σ is a closed submanifold of a compact manifold $\mathcal{M}$, then for r sufficiently small, there exists a finite number of Lipschitz open sets $(\Omega_i)_{1\le i\le N_0}$ such that

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for} \quad i \neq j$$

and

$$\overline{\Sigma}_r = \bigsqcup_{i=1}^{N_0} \overline{\Omega}_i. \tag{2.7}$$

We choose the open sets Ω_i, using the above Fermi coordinates, so that

$$\Omega_i = F_M^{p_i}\big(B^{N-k}(0,r) \times D_i\big) \quad \text{with} \quad p_i \in \Sigma, \tag{2.8}$$

where the D_i's are Lipschitz disjoint open sets of $\mathbb{R}^k$ such that

$$\bigcup_{i=1}^{N_0} \overline{f^{p_i}(D_i)} = \Sigma. \tag{2.9}$$

For $p \in \mathcal{M}$, we denote by $\sigma(p)$ the orthogonal projection of p on Σ. For the rest of the paper, if there is no confusion, we use the notation v_a instead of $v_{a,q}$. We get the following

Lemma 2.1 *Let $a \in \mathbb{R}$ and define*

$$v_{a,q}(p) = \big(-log\rho(p)\big)^a \rho(p)^\alpha \tag{2.10}$$

where for $x = F^{-1}(p) \in \mathbb{R}^N$

$$\alpha(x) = \frac{2+k-N}{2}\left(1 - \sqrt{1 - q(\sigma(x)) + |x|}\right). \tag{2.11}$$

Then we have

$$\begin{aligned} \Delta_g v_{a,q} = &-\big(\frac{N-k-2}{2}\big)^2 q\rho^{-2} v_{a,q} + a(a-1)\rho^{-2}\big(log\rho\big)^{-2} v_{a,q} \\ &+ \big(N-k-a\big)\rho^{-2}\big(log\rho\big)^{-1} v_{a,q} + O\big(log\rho\rho^{-3/2}(-log\rho)^a\rho^\alpha\big) \qquad \textit{in } \Sigma_r. \end{aligned} \tag{2.12}$$

Proof. If there is no ambiguity, we will write ω_a and v_a instead of $\omega_{a,q}$ and $v_{a,q}$, where

$$X_a(x) = \big(-log|x|\big)^a, \qquad \omega(x) = |x|^{\alpha(x)} \quad \text{and} \quad \omega_a = X_a\omega. \tag{2.13}$$

We can verify easily that

$$\Delta_{\mathbb{R}^N} w_a = X_a \Delta_{\mathbb{R}^N}\omega + 2\nabla X_a \nabla\omega + \omega\Delta_{\mathbb{R}^N} X_a. \tag{2.14}$$

We are going to calculate term by term the expression (2.14) using simple calculations.
We have that

$$\Delta\omega = \Delta(\varphi \circ u(x)),$$

where $\varphi(t) = e^t$ and

$$u(x) = \alpha(x)log(|x|) = logw.$$

But

$$\Delta(\varphi \circ u(x)) = \varphi''(u(x))|\nabla u(x)|^2 + \varphi'(u(x))\Delta u(x)$$

and

$$\varphi(u(x)) = \varphi'(u(x))$$

so that

$$\Delta\omega = \omega\Bigg[|\nabla \text{log}\omega|^2 + \Delta\text{log}\omega\Bigg]. \tag{2.15}$$

Since $\text{log}w = \alpha(x)(\text{log}|x|)$, we have that

$$\Delta\text{log}w = \alpha\Delta log|x| + 2\nabla\alpha\nabla(\text{log}|x|) + \text{log}|x|\Delta\alpha. \tag{2.16}$$

Using (2.15), we get

$$\Delta\alpha(x) = \alpha\Bigg[\frac{1}{2}\Delta\text{log}\big(1 - q(\sigma(x)) + |x|\big) + \frac{1}{4}|\nabla\text{log}\big(1 - q(\sigma(x)) + |x|\big)|^2\Bigg]. \tag{2.17}$$

But

$$\nabla\Bigg(\text{log}(1 - q(\sigma(x)) + |x|\Bigg) = \frac{-\nabla q(\sigma(x)) + \nabla|x|}{1 - q(\sigma(x)) + |x|}$$

and

$$\begin{aligned}
\Delta\text{log}(1 - q(\sigma(x)) + |x|) \quad &= \frac{\Delta(1 - q(\sigma(x)) + |x|)}{1 - q(\sigma(x)) + |x|} - \frac{|\nabla\Bigg(\text{log}(1 - q(\sigma(x)) + |x|\Bigg)|^2}{(1 - q(\sigma(x)) + |x|)^2} \\
&= \frac{\Delta(q \circ \sigma(x)) + \Delta|x|}{1 - q \circ \sigma(x) + |x|} - \frac{|\nabla\Bigg(\text{log}(1 - q(\sigma(x)) + |x|\Bigg)|^2}{(1 - q(\sigma(x)) + |x|)^2} \\
&= \frac{\Delta(q \circ \sigma(x)) + \Delta|x|}{1 - q \circ \sigma(x) + |x|} - \frac{|\nabla q \circ \sigma(x)|^2 + 1 - 2\nabla|x|\nabla(q \circ \sigma(x))}{\big(1 - q(\sigma(x)) + |x|\big)^2}.
\end{aligned}$$

Puting the above in (2.17), we obtain that

$$\Delta\alpha = \alpha\Big[\frac{1}{2}\quad\frac{-\Delta(q\circ\sigma(x))+\Delta|x|}{1-q\circ\sigma(x)+|x|} - \frac{1}{2}\quad\frac{|\nabla q\circ\sigma(x)|^2+1-2\nabla|x|\nabla(q\circ\sigma(x))}{\big(1-q\circ\sigma(x)+|x|\big)^2}$$
$$+\frac{1}{4}\quad\frac{|\nabla(q\circ\sigma(x))|^2+1-2\nabla|x|\nabla q\circ\sigma(x)}{\big(1-q\circ\sigma(x)+|x|\big)^2}\Big]. \tag{2.18}$$

Using the fact that $q\in\mathcal{C}^2$, we conclude that

$$\Delta\alpha(x) = O\big(|x|^{-3/2}\big). \tag{2.19}$$

We have also that

$$\begin{aligned}\nabla\alpha &= \Big(\frac{N-k-2}{2}\nabla\sqrt{1-q\circ\sigma(x)+|x|}\Big)\\ &= \frac{N-k-2}{2}\frac{1}{2\sqrt{1-q\circ\sigma(x)+|x|}}\nabla(1-q\circ\sigma(x)+|x|).\end{aligned}$$

Therefore

$$\nabla\alpha\nabla|x| = \frac{N-k-2}{4\sqrt{1-q\circ\sigma(x)+|x|}}\big(1-\nabla|x|\nabla(q\circ\sigma(x))\big).$$

Hence

$$\nabla\alpha\nabla|x| = O(|x|^{-1/2})$$

and from which we deduce that

$$\nabla\alpha\nabla(\log|x|) = \nabla\alpha\frac{\nabla|x|}{|x|} = O(|x|^{-3/2}). \tag{2.20}$$

Now let us evaluate the term $\Delta(\log\omega)$. We have that

$$\Delta(\log|x|) = \frac{N-k-2}{|x|^2} \tag{2.21}$$

so that

$$\alpha\Delta\log|x| = \alpha\frac{N-k-2}{|x|^2}.$$

Recall that

$$\Delta(\log\omega) = \log(|x|)\Delta\alpha + 2\nabla\alpha\nabla(\log|x|) + \alpha\Delta(\log(|x|)),$$

therefore

$$\Delta\log w = \alpha\frac{N-k-2}{2} + O(|x|^{-3/2}) + O(|x|^{-3/2}\log|x|)$$

and that

$$\Delta\log w = \alpha\frac{N-k-2}{|x|^2}\big(1+O(|x|)\big)+O\big(|x|^{-3/2}\big).$$

We have also that

$$\nabla(\log w) = \nabla(\alpha \log|x|) = \alpha \frac{\nabla|x|}{|x|} + \log|x| \nabla\alpha$$

and thus

$$|\nabla \log w|^2 = \frac{\alpha^2}{|x|^2} + (log|x|)^2 |\nabla\alpha|^2 + \frac{2\alpha}{|x|} \log|x| \nabla|x| \nabla\alpha = \frac{\alpha^2}{|x|^2} + O\big(\log|x||x|^{-3/2}\big).$$

Therefore (2.15) becomes

$$\frac{\Delta\omega}{\omega} = \alpha \frac{N-k-2}{|x|^2} + \frac{\alpha^2}{|x|^2} + O\big(\log|x||x|^{-3/2}\big). \tag{2.22}$$

We recall that, we want to calculate the Laplacian of ω_a define in (2.13). Then we have

$$\Delta\omega_a = \omega \Delta X_a + 2\nabla X_a \nabla\omega + X_a \Delta\omega.$$

From (2.22) we have that

$$X_a \Delta\omega = \omega_a \bigg[\frac{N-k-2}{|x|^2} + \frac{\alpha^2}{|x|^2} + O\big(\log|x||x|^{-3/2}\big) \bigg]. \tag{2.23}$$

Now we are going to evaluate $\omega \Delta X_a$. We have

$$\Delta X_a = \Delta(\varphi \circ u(x)),$$

where $\varphi(t) = (-logt)^a$ and $u(x) = |x|$. It's easy to verify that

$$\Delta X_a = \varphi''(u(x)) + \frac{N-k-1}{|x|}.$$

Therefore

$$\omega \Delta X_a = \omega_a \bigg[\frac{a(a-1)}{|x|^2 (log|x|)^2} + \frac{N-k-2}{|x|^2 (\log|x|)} a \bigg]. \tag{2.24}$$

Now let us finish this part by calculate the expression $2\nabla X_a \nabla\omega$. By simple calculations we get that

$$\nabla X_a = \bigg(\frac{a\nabla|x|}{|x| \log|x|} \bigg) X_a \tag{2.25}$$

and

$$\nabla\omega = \omega \bigg[\log|x| \nabla\alpha + \alpha \frac{\nabla|x|}{|x|} \bigg] \tag{2.26}$$

Therefore, using (2.25) and (2.26) we get that

$$\nabla X_a \nabla\omega = \omega_a \frac{a\nabla|x|}{|x| log|x|} \bigg(\log|x| \nabla\alpha + \alpha \frac{\nabla|x|}{|x|} \bigg) = \omega_a \bigg[\frac{a\nabla|x|\nabla\alpha}{|x|} + \frac{a\alpha}{|x|^2 \log|x|} \bigg]. \tag{2.27}$$

Then we conclude that

$$2\nabla X_a \nabla \omega = \omega_a \left[\frac{2a\nabla|x|\nabla\alpha}{|x|} + \frac{2a\alpha}{|x|^2 log|x|} \right]. \tag{2.28}$$

The sum of (2.23), (2.24) and (2.28) we get that
(2.29)

$$\Delta\omega_a = \omega_a \left[\alpha\frac{N-K-2}{|x|^2} + \frac{\alpha^2}{|x|^2} + \frac{a(a-1)}{|x|^2(\log|x|)^2} + a\frac{N-k-2}{|x|^2 log|x|} + \frac{2a\nabla|x|\nabla\alpha}{|x|} + \frac{2a\alpha}{|x|^2\log|x|} + O\big(\log|x||x|^{-3/2}\big) \right].$$

Moreover

$$\alpha(N-k-2) + \alpha^2 = \left(\frac{N-k-2}{2}\right)^2 \Big(-q\circ\sigma(x) + |x|\Big).$$

Then using (2.20) we can conclude that

$$\Delta\omega_a = -\left(\frac{N-k-2}{2}\right)^2 q|x|^{-2}\omega_a + a(a-1)|x|^{-2}\big(\log|x|\big)^{-2}\omega_a \tag{2.30}$$

$$+\big(N-k-a\big)|x|^{-2}\big(\log|x|\big)^{-1}\omega_a + O\big(\log|x||x|^{-3/2}\big)\omega_a.$$

Using the Laplace-Beltrami operator

$$\Delta_g = -g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial}{\partial x_k}\right), \tag{2.31}$$

and the aproximations

$$\Gamma^k_{ij}(x) = O_k(|y|)$$

and

$$g_{ij}(x) = \delta_{ij} + O(|y|^2),$$

it follows that

$$\Delta_g v_a = \Delta_{\mathbb{R}^N}\omega_a(F(x)) + O_{ij}\big(\rho^2\big)\partial_{ij}\omega_a + O_k(\rho)\partial_k\omega_a. \tag{2.32}$$

Now using the above identity, we conclud that for ρ_{Σ_k} small enough

$$\Delta_g v_a = -\left(\frac{N-k-2}{2}\right)^2 q\rho^{-2}v_a + a(a-1)\rho^{-2}\big(\log\rho\big)^{-2}v_a \tag{2.33}$$

$$+\big(N-k-a\big)\rho^{-2}\big(\log\rho\big)^{-1}v_a + O\big((\log\rho)\rho^{-3/2}\big)v_a \qquad \text{in } \Sigma_r.$$

□

2.1 Construction of a sub and supersolutions

For $\lambda \in \mathbb{R}$, $\eta \in Lip(\mathcal{M})$ with $\eta = 0$ on Σ and $q \in \mathcal{C}^2(\mathcal{M})$, $q > 0$ in $\mathcal{M}$ with $\max_{\Sigma} q(\sigma) = 1$, we define the operator

$$L_\lambda := -\Delta - \left(\frac{N-k-2}{2}\right)^2 q\rho^{-2} + \lambda\eta\rho^{-2}. \tag{2.34}$$

Using Lemma 2.1 and (2.34) it's easy to verify that

$$Lv_a = -a(a-1)\rho^{-2}\big(\log\rho\big)^{-2}v_a - \big(N-k-a\big)\rho^{-2}(\log\rho)^{-1}v_a + \lambda\eta\rho^{-2}v_a + O\big(\log\rho)\rho^{-3/2}\big)v_a. \tag{2.35}$$

In this subsection we wish to construct a subsolution and a supersolution for the operator L_λ defined above. For that we obtain the following lemmas

Lemma 2.2 *There exists r_0 such that for all $r \in (0, r_0)$ and for all $\epsilon \in [0,1)$, the function*

$$V_\epsilon = v_{-1,q} + v_{0,q-\epsilon} \tag{2.36}$$

satisfies

$$L_\lambda V_\epsilon \le 0 \textit{ in } \Sigma_r, \quad \textit{for all} \quad \epsilon \in [0,1). \tag{2.37}$$

Moreover $V_\epsilon \in H^1(\Sigma_r)$ for any $\epsilon \in (0,1)$ and in addition

$$\int_{\Sigma_r} V_0^2\rho^{-2}dv_g \ge C\int_{\Sigma}\frac{1}{\sqrt{1-q(\sigma)}}d\sigma \tag{2.38}$$

Proof. Using polar coordinates it's easy to see that $v_a \in H^1(\Sigma_r)$ for all a such that $a < -\dfrac{1}{2}$ and that $v_{0,q-\varepsilon} \in H^1(\Sigma_r)$ for all $\varepsilon > 0$. We therefore skip the proof. Now we have that for $a = -1$ and for r small enough

$$L_\lambda v_{-1,q} = -2\rho^{-2}\big(\log\rho\big)^{-2}v_{-1,q} + \lambda\eta\rho^{-2}v_{-1,q} - \big(N-k+1\big)\rho^{-2}(\log\rho)^{-1}v_{-1} + O\big(\rho^{-3/2}(\log\rho)v_{-1,q}\big). \tag{2.39}$$

Therefore

$$L_\lambda v_{-1,q} \le \left[-2\rho^{-2}\big(\log\rho\big)^{-2} + C|\log\rho|\rho^{-3/2} + |\lambda|\eta\rho^{-2}\right]v_{-1,q} \qquad \text{in } \Sigma_r. \tag{2.40}$$

Using the fact that $\eta = 0$ on Σ and $\eta \in Lip(M)$ we have $|\eta| < C\rho$ around Σ. Therefore

$$L_\lambda v_{-1,q} \leq -\rho^{-2}(\log\rho)^{-2} v_{-1,q} = \rho^{-2}(\log\rho)^{-3} v_{0,q} \qquad \text{in } \Sigma_r. \tag{2.41}$$

Using the same arguments as above, we get that

$$L_\lambda v_{0,q-\epsilon} \leq C|\log\rho|\rho^{-3/2} v_{0,q-\epsilon} \qquad \text{in } \Sigma_r \quad \forall\epsilon \in [0,1). \tag{2.42}$$

Therefore using (2.41) and (2.42) we get (2.37). (2.36) implies that

$$\begin{aligned}\int_{\Sigma_r} \frac{V_0^2}{\rho^2} dv_g &\geq \int_{\Sigma_r} \frac{v_{0,q}^2}{\rho^2} dv_g \\ &= \int_{\Sigma_r} \rho^{2\alpha-2} dv_g.\end{aligned}$$

Using (2.7), we get that

$$\int_{\Sigma_r} \frac{V_0^2}{\rho^2} dv_g \geq \int_{\bigcup_{i=1}^{N_0} \overline{\Omega}_i} \rho^{2\alpha-2} dv_g,$$

$$\begin{aligned}\int_{\Sigma_r} \frac{V_0^2}{\rho^2} dv_g &\geq \textstyle\sum_{i=1}^{N_0} \displaystyle\int_{\Omega_i} \rho^{2\alpha-2} dv_g \\ &= \textstyle\sum_{i=1}^{N_0} \displaystyle\int_{F_{\mathcal{M}}^{p_i}(B^{N-k}(0,r)\times D_i)} \rho^{2\alpha-2} dv_g.\end{aligned}$$

Using change of variable formula we get

$$\int_{\Sigma_r} \frac{V_0^2}{\rho^2} dv_g \geq \sum_{i=1}^{N_0} \int_{B^{N-k}(0,r)\times D_i} |z|^{2\alpha(F_{\mathcal{M}}^{p_i}(x))-2} |\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(x) dx.$$

Notice that $|\mathrm{Jac}(F_{\mathcal{M}}^{p_i})|(x)$ is bounded, the function $|z|^{-\sqrt{|z|}}$ is also bounded in a neighborhood of the ball centered at 0. Moreover

$$\alpha(F_{\mathcal{M}}^{p_i}(x)) = (2+k-N)(1 - \sqrt{1 - q(f^{p_i}(y)) + |z|}) \tag{2.43}$$

so that

$$\begin{aligned}\int_{\Sigma_r} \frac{V_0^2}{\rho^2} dv_g &\geq C \textstyle\sum_{i=1}^{N_0} \displaystyle\int_{B^{N-k}(0,r)\times D_i} |z|^{k-N} |z|^{-(2+k-N)\sqrt{1-q(f^{p_i}(y))}} |z|^{-\sqrt{|z|}} dx \\ &\geq C \textstyle\sum_{i=1}^{N_0} \displaystyle\int_{B^{N-k}(0,r)\times D_i} |z|^{k-N} |z|^{-(2+k-N)\sqrt{1-q(f^{p_i}(y))}} dx.\end{aligned}$$

Using polar coordinates, we get

$$\int_{\Sigma_r}\frac{V_0^2}{\rho^2}dv_g \geq C\sum_{i=1}^{N_0}\int_{D_i}\int_{S^{N-k-1}}d\theta\int_0^r t^{N-k-1}t^{k-N}t^{(N-k-2)\sqrt{1-q(f^{p_i}(y))}}dtdy$$

$$\geq C\sum_{i=1}^{N_0}\int_{D_i}\int_0^{r_{i_1}} t^{-1}t^{(N-k-2)\sqrt{1-q(f^{p_i}(y))}}|\mathrm{Jac}(f^{p_i})|(y)dy.$$

Therefore, using the fact that $|\mathrm{Jac}(f^{p_i})|(y) = 1 + O(r)$ and so bounded, we get the result

$$\int_{\Sigma_r}\frac{V_0^2}{\rho^2}dv_g \geq C\int_{\Sigma}\int_0^r t^{-1+(N-k-2)\sqrt{1-q(\sigma)}}drd\sigma$$

(2.44)
$$\geq C\int_{\Sigma}\frac{r^{(N-k-2)\sqrt{1-q(\sigma)}}}{(N-k-2)\sqrt{1-q(\sigma)}}d\sigma$$

$$\geq C\int_{\Sigma}\frac{d\sigma}{\sqrt{1-q(\sigma)}}.$$

This ends the proof. □

Lemma 2.3 *there exists r_0 such that for all $r\in(0,r_0)$ the function*

(2.45)
$$U = v_0 - v_{-1} > 0 \qquad \textit{in } \Sigma_r$$

and satisfies $L_\lambda U \geq 0$ in Σ_r. Moreover $U\in H^1(\Sigma_r)$ provided

(2.46)
$$\int_{\Sigma}\frac{d\sigma}{\sqrt{1-q(\sigma)}} < +\infty.$$

Proof. Using (2.34), we have that

(2.47)
$$L_\lambda v_0 = -\big(N-k\big)\rho^{-2}\big(\log\rho\big)^{-2}v_0 + \lambda\eta\rho^{-2}v_0 + O\big((\log\rho)\rho^{-3/2}\big)v_0$$

and

(2.48)
$$-L_{v_{-1}} = 2\rho^{-2}\big(\log\rho\big)^{-2}v_{-1}+(N-k+1)\rho^{-2}\big(\log\rho\big)^{-1}v_{-1}-\lambda\eta\rho^{-2}v_{-1}+O\big((\log\rho)\rho^{-3/2}\big)v_{-1}.$$

so that

(2.49) $$L_\lambda v_0 \geq -|\lambda|\eta\rho^{-2}v_0 - C|\text{log}\rho)|\rho^{-3/2}v_0,$$

(2.50) $$L_\lambda v_{-1} \geq \big(2\rho^{-2}(\text{log}\rho)^{-2} - C|\text{log}\rho|\rho^{-3/2} - |\lambda|\eta\rho^{-2}\big)v_{-1}.$$

The dominant term in the right hand sides of the two above inequalities is $2\rho^{-2}(log\rho)^{-2}$. Therefore there exists r_0 small such that for all $r \in (0, r_0)$ the inequality

(2.51) $$L_\lambda U \geq 0 \text{ in } \Sigma_r$$

holds. Now we prove that $U \in H^1(\Sigma_r)$ provided inequality (2.46) holds. We have that

$$\nabla_g v_0 = \nabla(\rho^\alpha) = v_0\nabla(\alpha\text{log}\rho) = v_0\bigg(\text{log}\rho\nabla\alpha + \alpha\frac{\nabla\rho}{\rho}\bigg).$$

Hence

$$|\nabla v_0|^2 = v_0^2\bigg[|\text{log}\rho\nabla\alpha|^2 + \alpha^2\frac{|\nabla\rho|^2}{\rho^2} + 2\alpha(\text{log}\rho)\frac{\nabla\alpha\nabla\rho}{\rho}\bigg].$$

Using the fact that α is of class $\mathcal{C}^1$ and the estimation

$$2\alpha(log\rho)\frac{\nabla\alpha\nabla\rho}{\rho} = O(\rho^{-2}log\rho)$$

we deduce that there exists a positive constant C such that

$$|\nabla v_0|^2 \leq Cv_0^2\rho^{-2} = C\rho^{2\alpha-2}.$$

Therefore

$$\int_{\Sigma_r}|\nabla v_0|^2 dv_g \leq C\int_{\Sigma_r}\rho^{2\alpha-2}dv_g.$$

As in the above lemma and using polar coordinates, we get

$$\int_{\Sigma_r}|\nabla v_0|^2 dv_g \leq C\sum_{i=1}^{N_0}\int_{D_i}\int_{S^{N-k-1}}d\theta\int_0^r t^{-1}t^{(N-k-2)\sqrt{1-q(f^{p_i}(y))}}dtdy.$$

Also as in the above lemma

$$\sum_{i=1}^{N_0}\int_{D_i}\frac{1}{\sqrt{1-q(f^{p_i}(y))}}dy \leq C\int_\Sigma\frac{1}{\sqrt{1-q(\sigma)}}d\sigma$$

so that

$$\int_{\Sigma_r} |\nabla v_0|^2 dv_g \leq C \int_{\Sigma} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma. \tag{2.52}$$

This ends the proof of the Lemma. □

3 Proof of Theorem 1.1

In this section we give a complete proof of Theorem 1.1. In the first subsection we prove the existence of λ^* verifying (1.8) and (1.9). In the second and last one of this section we give the proof of the existence and non-existence result for $\lambda \neq \lambda^*$.

3.1 Existence of λ^*

Lemma 3.1 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$ and let Σ be a closed submanifold of dimension $1 \leq k \leq N-2$. We assume that the weight functions b, q and η satisfy* (1.6) *and* (1.7)*. Then there exists $r_0 > 0$ and $C > 0$ depending only on $\mathcal{M}, \Sigma, q, \eta$ and b such that for all $r \in (0, r_0)$ the inequality*

$$\int_{\Sigma_r} b|\nabla u|^2 dv_g \geq \left(\frac{N-k-2}{2}\right)^2 \int_{\Sigma_r} q\frac{|u|^2}{\rho^2} dv_g + C\int_{\Sigma_r} \frac{|u|^2}{\rho^2(log\rho)^2} dv_g \tag{3.1}$$

holds for all $u \in H^1(\Sigma_r)$.

Proof. We have that $\dfrac{b}{q} \in \mathcal{C}^2(\mathcal{M})$, there exists $C > 0$ such that:

$$\left|\frac{b(p)}{q(p)} - \frac{b(\sigma(p))}{q(\sigma(p))}\right| < C\rho, \quad \forall \ p \in \Sigma_r \tag{3.2}$$

for r small enough. Hence by (1.7), there exists $C' > 0$ such that

$$b(p) \geq q(p) - C'\rho, \quad \forall p \in \Sigma_r. \tag{3.3}$$

Let $V = v_{1/2,q}$ in Σ_r. We have that

$$\text{div}(b\nabla V) = b\Delta V + \nabla p \nabla V.$$

and by lemma (2.1) we get

$$-\frac{\text{div}(b\nabla V)}{V} \geq b\left(\frac{N-k-2}{2}\right)^2 + \frac{1}{4} b\rho^{-2}(\text{log}\rho)^{-2} + O(\rho^{-3/2}|\text{log}\rho|) \text{ in } \Sigma_r. \tag{3.4}$$

Using (3.3) with the above inequality we get

$$-\frac{\text{div}(b\nabla V)}{V} \geq \left(\frac{N-k-2}{2}\right)^2 q\rho^{-2} + c\rho^{-2}(\text{log}\rho)^{-2} \text{ in } \Sigma_r, \tag{3.5}$$

where c is a positive constant depending only on $\mathcal{M}, \Sigma, q, \eta$ and b.
Let $u \in \mathcal{C}_c^\infty(\mathcal{M} \setminus \Sigma)$ and define

$$\varphi := \frac{u}{V}.$$

Then we have that

$$b|\nabla u|^2 = b\big(|V\nabla\varphi| + \nabla V \nabla(V\varphi^2)\big). \tag{3.6}$$

Hence by integration by parts we get that

$$\int_{\Sigma_r} |\nabla u|^2 b dv_g = \int_{\Sigma_r} |V\nabla\varphi|^2 dv_g + \int_{\Sigma_r} \left(-\frac{\text{div}(p\nabla V)}{V}\right) u^2 dv_g.$$

Using (3.5), we get

$$\int_{\Sigma_r} b|\nabla u|^2 dv_g \geq \left(\frac{N-k-2}{2}\right)^2 \int_{\Sigma_r} q\frac{|u|^2}{\rho^2} dv_g + C\int_{\Sigma_r} \frac{|u|^2}{\rho^2(\text{log}\rho)^2} dv_g \tag{3.7}$$

for all $u \in \mathcal{C}_c^\infty(\Sigma_r)$. Using the fact that $\mathcal{C}_c^\infty(\mathcal{M} \setminus \Sigma)$ is dense in $\mathcal{C}_c^\infty(\mathcal{M})$ the proof remain valid for a general u. Furtheremore $\mathcal{C}_c^\infty(\Sigma_r)$ is dense in $H^1(\Sigma_r)$. This ends the proof. □

Lemma 3.2 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$ and let Σ be a closed submanifold of dimension $1 \leq k \leq N$. Assume that* (1.6) *and* (1.7) *hold. Then there exists $\lambda^* = \lambda^*(\mathcal{M}, \Sigma, b, q, \eta) \in \mathbb{R}$ such that*

$$\mu_\lambda(\mathcal{M}, \Sigma) = \left(\frac{N-k-2}{2}\right)^2, \quad \forall \lambda \leq \lambda^*,$$

$$\mu_\lambda(\mathcal{M}, \Sigma) < \left(\frac{N-k-2}{2}\right)^2, \quad \forall \lambda > \lambda^*.$$

Proof. For $b = q = 1$ and $\eta = \rho^2$, we define $\nu_\lambda(\mathcal{M}, \Sigma) := \mu_\lambda(\mathcal{M}, \Sigma)$. It's known that

$$\nu_\lambda(\mathbb{R}^N, \mathbb{R}^k) = \left(\frac{N-k-2}{2}\right)^2. \tag{3.8}$$

Therefore for any $\delta > 0$, we can find $u_\delta \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u_\delta|^2 dx \leq \left(\left(\frac{N-k-2}{2}\right)^2 + \delta\right) \int_{\mathbb{R}^N} |z|^{-2} u_\delta^2 dx, \tag{3.9}$$

where $x = (z, y) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$. By (1.1) there exists $\sigma_0 \in \Sigma$ such that $b(\sigma_0) = q(\sigma_0)$. For $r > 0$, we let $\rho_r > 0$ such that for all $p \in B(\sigma_0, \rho_r)$ we have the following

$$\begin{cases} b(p) \leq (1+r) q(\sigma_0) \\ q(p) \geq (1-r) q(\sigma_0) \\ \eta(p) \leq r. \end{cases} \tag{3.10}$$

Let $\epsilon_0 > 0$ small such that for all $\varepsilon \in (0, \varepsilon_0)$, $F_{\mathcal{M}}^{\sigma_0}(\varepsilon \text{supp}(u_\delta)) \subset B(\sigma_0, \rho_r)$ and we let

$$x = \varepsilon^{\frac{2-N}{2}} F^{-1}(p)).$$

Therefore we define

$$v(p) = \varepsilon^{\frac{2-N}{2}} u_\delta(\varepsilon^{-1} F^{-1}(p)).$$

It's clear that for every $\varepsilon \in (0, \varepsilon_0)$, $v \in \mathcal{C}_c^\infty(\mathcal{M})$. By applying the change of variable formula and (3.10), we get

$$\begin{aligned}
\mu_\lambda(\mathcal{M}, \Sigma) &\leq \frac{\displaystyle\int_{\mathcal{M}} b|\nabla v|^2 dv_g + \lambda \int_{\mathcal{M}} \rho^{-2} \eta v^2 dvg}{\displaystyle\int_{\mathcal{M}} q\rho^{-2} v^2 dv_g} \\
&\leq \frac{1+r}{1-r} \frac{\displaystyle\int_{\mathcal{M}} |\nabla v|^2 dv_g}{\displaystyle\int_{\mathcal{M}} \rho^{-2} v^2 dv_g} + \frac{|\lambda| r}{(1-r) q(\sigma_0)} \\
&\leq \frac{(1+r)(1+C\varepsilon)}{(1-r)(1-C\varepsilon)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_\delta|^2 dx}{\displaystyle\int_{\mathcal{M}} |z|^{-2} u_\delta^2 dx} + \frac{|\lambda| r}{(1-r) q(\sigma_0)} \\
&\leq \big(1 + O(r)\big)(1 + O(\varepsilon))\left((\frac{N-k-2}{2})^2 + \delta \right) + O(r).
\end{aligned}$$

As $\varepsilon, r, \delta \to 0$ respectively, we get that

$$\mu_\lambda(\mathcal{M}, \Sigma) \leq \left(\frac{N-k-2}{2} \right)^2, \quad \forall \lambda \in \mathbb{R}. \tag{3.11}$$

To finish the proof of the lemma we have just to show the existence of $\bar{\lambda} \in \mathbb{R}$ such that

$$\mu_{\bar{\lambda}}(\mathcal{M}, \Sigma) \geq (\frac{N-k-2}{2})^2.$$

Indeed we let $\varphi \in \mathcal{C}_c^\infty(\mathcal{M})$ such that

$$\varphi = \begin{cases} 1, & \text{in } \Sigma_r \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

For $u \in H^1(\mathcal{M})$, we write

$$u = u\varphi + (1 - \varphi) u$$

and notice that

$$u\varphi \in H^1(\Sigma_r).$$

We then have that

$$\begin{aligned}\int_{\mathcal{M}} u^2\rho^{-2}q dv_g &= \int_{\mathcal{M}} |u\varphi + (1-\varphi)u|^2\rho^{-2}q dv_g \\ &= \int_{\mathcal{M}} |u\varphi|^2\rho^{-2}q dv_g + \int_{\mathcal{M}} |(1-\varphi)u|^2\rho^{-2}q dv_g + 2\int_{\mathcal{M}} |u^2\varphi(1-\varphi)|q\rho^{-2} dv_g \\ &\le \int_{\Sigma_r} |u\varphi|^2\rho^{-2}q dv_g + 3\int_{\Sigma_r^c} |(1-\varphi)|u^2\rho^{-2}q dv_g\end{aligned}$$

Using Lemma 3.1, we have that

$$\int_{\Sigma_r} |u\varphi|^2\rho^{-2}q dv_g \le \left(\frac{N-k-2}{2}\right)^{-2}\int_{\mathcal{M}} b|\nabla u|^2 dv_g. \tag{3.13}$$

Therefore, there exists $C > 0$ such that

$$\int_{\mathcal{M}} qu^2\rho^{-2} dv_g \le \left(\frac{N-k-2}{2}\right)^{-2}\int_{\mathcal{M}} b|\nabla u|^2 dv_g + C\int_{\mathcal{M}} \rho^{-2}u^2\eta dv_g, \quad \forall u \in \mathcal{C}_c^\infty(\mathcal{M}).$$

Taking $\bar{\lambda} = -C$ we get the result. Since the function $\lambda \to \mu_\lambda(\mathcal{M},\Sigma)$ is decreasing, we can define λ^* as

$$\lambda^* := \sup\left\{\lambda \in \mathbb{R} : \mu_\lambda(\mathcal{M},\Sigma) = \left(\frac{N-k-2}{2}\right)^2\right\}. \tag{3.14}$$

this ends the proof or the Lemma. $\square$

3.2 Existence and non-existence result in the case $\lambda \neq \lambda^*$

Theorem 3.3 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension N and let Σ be a closed submanifold of dimension $1 \le k \le N-2$. We assume that the weight functions b, q and η verify* (1.6) *and* (1.7)*. Then $\mu_\lambda(\mathcal{M},\Sigma)$ is not achieved for every $\lambda < \lambda^*$.*

Proof. We suppose by contradiction that for some $\lambda_1 < \lambda^*$ the infinimum $\mu_\lambda(\mathcal{M},\Sigma)$ is attained at an element $u_1 \in H^1(\mathcal{M}\setminus\Sigma)$. We suppose that u_1 is normalised so that

$$\int_{\mathcal{M}} \rho^{-2}|u_1|^2 q dv_g = 1$$

and

$$\int_{\mathcal{M}} b|\nabla u_1|^2 dv_g - \lambda_1 \int_{\mathcal{M}} \rho^{-2}|u_1|^2 \eta dv_g = \left(\frac{N-k-2}{2}\right)^2.$$

Then for $\lambda_1 < \lambda < \lambda^*$, we have that

(3.15)

$$\left(\frac{N-k-2}{2}\right)^2 = \mu_\lambda(\mathcal{M}, \Sigma) \leq \int_{\mathcal{M}} b|\nabla u_1|^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2}|u_1|^2 \eta dv_g < \left(\frac{N-k-2}{2}\right)^2,$$

which is impossible. So for any $\lambda < \lambda^*$, $\mu_\lambda(\mathcal{M}, \Sigma)$ is not achieved. This ends the proof of the theorem. □

Theorem 3.4 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension N and let Σ_k be a closed submanifold of dimension $1 \leq k \leq N-2$. We assume that the weight functions b, q and η verify* (1.6) *and* (1.7)*. Then $\mu_\lambda(\mathcal{M}, \Sigma)$ is achieved for every $\lambda > \lambda^*$.*

Proof. A similar proof was done by Thiam in [53]. So we expose here a similar one. Let $\{u_n\}$ be a minimizing sequence of $\mu_\lambda(\mathcal{M}, \Sigma)$ normalized so that

$$\int_{\mathcal{M}} \rho^{-2} u_n^2 q dv_g = 1.$$

So we have that

$$\mu_\lambda(\mathcal{M}, \Sigma) + o(1) = \int_{\mathcal{M}} b|\nabla u_n|^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2} u_n^2 \eta dv_g. \tag{3.16}$$

Thus $\{u_n\}$ is bounded in $H^1(\mathcal{M})$. After passing to a subsequence, we may assume that there exists $u \in H^1(\mathcal{M})$ such that

$$v_n = u_n - u \rightharpoonup 0 \text{ in } H^1(\mathcal{M}), \;\; v_n \longrightarrow 0 \text{ in } L^2(\mathcal{M}), \;\; v_n \rightharpoonup 0 \text{ in } H^1(\mathcal{M}), \tag{3.17}$$
$$\frac{v_n}{\rho}\sqrt{\eta} \to 0 \text{ in } L^2(\mathcal{M}) \text{ and } \frac{v_n\sqrt{q}}{\rho} \to 0 \text{ in } L^2(\mathcal{M}).$$

Using (3.16) and (3.17) we obtain that

(3.18)

$$\begin{aligned}\mu_\lambda(\mathcal{M}, \Sigma) &= \int_{\mathcal{M}} b|\nabla u_n|^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2} u_n^2 \eta dv_g + o(1) \\ &= \int_{\mathcal{M}} b|\nabla u|^2 dv_g + \int_{\mathcal{M}} b|\nabla v_n|^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2}|u|^2 \eta dv_g + o(1)\end{aligned}$$

and

$$1 = \int_{\mathcal{M}} \rho^{-2} u_n^2 q dv_g + o(1) = \int_{\mathcal{M}} \rho^{-2} u^2 q dv_g + \int_{\mathcal{M}} \rho^{-2} v_n^2 q dv_g + o(1). \tag{3.19}$$

Let $\lambda < \lambda^*$ so that

$$\int_{\mathcal{M}} b|\nabla v_n|^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2} v_n^2 \eta dv_g \geq \left(\frac{N-k-2}{2}\right)^2 \int_{\mathcal{M}} \rho^{-2} v_n^2 q dv_g + o(1).$$

Hence by (3.19) and (3.17)

$$\int_{\mathcal{M}} b|\nabla v_n|^2 dv_g \geq \left(\frac{N-k-2}{2}\right)^2 \left(1 - \int_{\mathcal{M}} u^2 \rho^{-2} q dv_g\right) + o(1). \tag{3.20}$$

By (3.18) and (3.20) we obtain that

$$\int_{\mathcal{M}} b|\nabla u|^2 dv_g + \left(\frac{N-k-2}{2}\right)^2 \left(1 - \int_{\mathcal{M}} u^2 \rho^{-2} q dv_g\right) - \lambda \int_{\mathcal{M}} \rho^{-2} u^2 \eta dv_g \geq \mu_\lambda(\mathcal{M}, \Sigma). \tag{3.21}$$

But

$$\int_{\mathcal{M}} b|\nabla u|^2 dv_g - \lambda \int_{\mathcal{M}} \rho^{-2} u^2 \eta dv_g \geq \mu_\lambda(\mathcal{M}, \Sigma) \int_{\mathcal{M}} u^2 \rho^{-2} q dv_g \tag{3.22}$$

so that

$$\left(\mu_\lambda(\mathcal{M}, \Sigma) - \left(\frac{N-k-2}{2}\right)^2\right) \left(\int_{\mathcal{M}} u^2 \rho^{-2} q dv_g - 1\right) \leq 0. \tag{3.23}$$

Since

$$\mu_\lambda(\mathcal{M}, \Sigma) < \left(\frac{N-k-2}{2}\right)^2,$$

we get that

$$1 \leq \int_{\mathcal{M}} u^2 \rho^{-2} q dv_g.$$

But by Fatou's Lemma

$$1 \geq \int_{\mathcal{M}} u^2 \rho^{-2} q dv_g.$$

Therefore

$$1 = \int_{\mathcal{M}} u^2 \rho^{-2} q dv_g. \tag{3.24}$$

We can conclude that u is a minimizer for $\mu_\lambda(\mathcal{M}, \Sigma)$ and

$$\int_{\mathcal{M}} b|\nabla v_n|^2 dv_g \longrightarrow 0.$$

Thus $u_n \longrightarrow u$ in $H^1(\mathcal{M})$ and the proof. $\square$

These two above results of this section represent a complete proof of Theorem 1.1.

4 Proof of theorem 1.2

In this section we give a complete proof of Theorem 1.2. For that we have the following results

Theorem 4.1 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$, Σ be a closed submanifold of dimension $1 \leq k \leq N-2$ and $\lambda \geq 0$. Assume that the weight functions b, q and η satisfy* (1.6) *and* (1.7)*. We supose also that $u \in H^1(\mathcal{M} \setminus \Sigma) \cap \mathcal{C}(\mathcal{M} \setminus \Sigma)$ is a non-negative solution satisfying*

$$(4.1) \qquad -\mathit{div}(b\nabla u) - \left(\frac{N-k-2}{2}\right)^2 q\rho^{-2}u \geq -\lambda\eta\rho^{-2}u \quad \mathit{in} \quad \mathcal{M}.$$

Moreover if

$$(4.2) \qquad \int_\Sigma \frac{1}{\sqrt{1-q(\sigma)/b(\sigma)}} d\sigma = +\infty$$

then $u \equiv 0$.

Proof. We assume by contradiction that u does not vanish identically near Σ and satisfies (4.1). Therefore by standard regularity and the maximum principle, see

[24], u is smooth and positive in Σ_r for some $r > 0$ small. Let $\overline{u} := \sqrt{b}u$ and then

$$\begin{aligned}
\Delta\overline{u} &= \Delta(\sqrt{b}u)\\
&= \sqrt{b}\Delta u + u\Delta\sqrt{b} + 2\nabla u\frac{\nabla b}{2\sqrt{b}}\\
&= \sqrt{b}\Delta u + u\sqrt{b}\left[\frac{1}{4}|\nabla \text{log} b|^2 + \frac{1}{2}\Delta\text{log} b\right] + \frac{\nabla b\nabla u}{\sqrt{b}}\\
&= \frac{1}{\sqrt{b}}\big(b\Delta u + \nabla b\nabla u\big) + u\sqrt{b}\left(\frac{|\nabla b|^2}{4b^2} - \frac{|\nabla b|^2}{2b^2} + \frac{\Delta b}{2b}\right)\\
&= \frac{1}{\sqrt{b}}\text{div}(b\nabla u) + u\sqrt{b}\left(\frac{|\nabla b|^2}{4b^2} - \frac{|\nabla b|^2}{2b^2} + \frac{\Delta b}{2b}\right).
\end{aligned}$$

Therefore using (4.1), we get that

$$(4.3)\quad -\Delta\overline{u} - \left(\frac{N-k-2}{2}\right)^2\frac{q}{b}\rho^{-2}\overline{u} \geq -\lambda\frac{\eta}{b}\rho^{-2}\overline{u} + \left(\frac{\Delta b}{2b} + \frac{|\nabla b|^2}{4b^2}\right)\overline{u} \quad \text{in} \quad \mathcal{M}.$$

Since $b \in \mathcal{C}^2(\mathcal{M})$ and $b > 0$ in $\mathcal{M}$, the result is the same as in the case $b \equiv 1$ and q/b replaced by q. See Brezis-Marcus [7] or Fall-Mahmoudhi [20]. So withoout lost of generality, we suppose that $b \equiv 1$ and consider the function $V_\varepsilon \in H^1(\Sigma_r)$ given by Lemma 2.2 which satisfies

$$(4.4)\qquad L_\lambda V_\varepsilon \leq 0 \text{ in } \Sigma_r, \text{ for all } \varepsilon \in (0,1).$$

According to (4.4) and (4.1), we let $R > 0$ such that

$$(4.5)\qquad RV_\varepsilon \leq u \text{ on } \partial\Sigma_r$$

and define

$$W_\varepsilon = RV_\varepsilon - u$$

so that $W_\varepsilon^+ \in H^1(\Sigma_r)$. Moreover by (4.1) and (4.4) we get that

$$(4.6)\qquad L_\lambda W_\varepsilon \leq 0 \text{ in } \Sigma_r, \quad \forall\varepsilon \in (0,1).$$

Multiplying the above inequality by W_ε^+ and integrating by pats we get

$$\int_{\Sigma_r} |\nabla W_\varepsilon^+|^2 dv_g - \left(\frac{N-k-2}{2}\right)^2 \int_{\Sigma_r} \rho^{-2} q |W_\varepsilon^+|^2 dv_g + \lambda \int_{\Sigma_r} \eta \rho^{-2} |W_\varepsilon^+|^2 dv_g \leq 0.$$

Then Lemma 3.1 implies that $W_\varepsilon^+ = 0$ in Σ_r provided r small enough because of the fact that $|\eta| \leq C\rho$ near Σ. Therefore $u \geq RV_\varepsilon$ for every $\varepsilon \in (0,1)$. In particular $u \geq RV_0$. Hence by Lemma 2.2, we have that

$$\infty > \int_{\Sigma_r} u^2 \rho^{-2} dv_g \geq R^2 \int_{\Sigma_r} V_0^2 \rho^{-2} dv_g \geq \int_\Sigma \frac{d\sigma}{\sqrt{1-q(\sigma)}}. \tag{4.7}$$

This is impossible because of (4.2). Therefore $u \equiv 0$ in Σ_r and by the maximum principle $u \equiv 0$ in $\mathcal{M}$. This ends the proof of the theorem.

Theorem 4.2 *Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension N and let Σ be a closed submanifold of dimension $1 \leq k \leq N-2$. We assume that the weight functions b, q and η verify* (1.6) *and* (1.7). *If*

$$\int_\Sigma \frac{1}{\sqrt{1 - b(\sigma)/q(\sigma)}} d\sigma < \infty \tag{4.8}$$

then $\mu_{\lambda^*} = \mu_{\lambda^*}(\mathcal{M}, \Sigma)$ is achieved.

Proof. As in Theorem 4.1 we supose without any loss of generality that $b \equiv 1$. Let $\{\lambda_n\}$ be a sequence of real numbers decreasing to λ^*. This means that $\lambda_n > \lambda^*$ for all $n \in \mathbb{N}$. By Theorem 3.3 , there exists $u_n \in H^1(\mathcal{M})$ such that for all $n \in \mathbb{N}$

$$-\Delta_g u_n - \mu_{\lambda_n}(\mathcal{M}) \rho^{-2} q u_n = -\lambda_n \rho^{-2} \eta u_n \qquad \text{in } \mathcal{M}. \tag{4.9}$$

Recall that for $u_n \in H^1(\mathcal{M})$, $|u_n| \in H^1(\mathcal{M})$ and $|\nabla u_n| = |\nabla |u_n||$. See for instance books [17] and [27] for mor details. Therefore we suppose that $u_n \geq 0$ in $\mathcal{M}$ and $||\rho^{-1} u_n||_2^2 = 1$. Hence

$$u_n \rightharpoonup u \qquad \text{in} \quad H^1(\mathcal{M} \setminus \Sigma) \quad \text{and} \quad u_n \longrightarrow u \qquad \text{in} \quad L^2(\mathcal{M}).$$

We have that

$$\Delta_g u_n + \big(\mu_{\lambda_n} \rho^{-2} q - \lambda_n \rho^{-2} \eta\big) u_n = 0 \qquad \text{in } \mathcal{M}. \tag{4.10}$$

We want to show that there exists $C > 0$ such that

$$\forall n \in \mathbb{N}, \quad u_n \leq CU \text{ in } \Sigma_r. \tag{4.11}$$

Indeed we can choose $C > 0$ such that

$$\forall n \in \mathbb{N}, \ \ v_n := u_n - CU \leq 0 \text{ on } \partial\Sigma_r.$$

It's clear that $v_n^+ \in H^1(\Sigma_r)$. Hence

$$L_{\lambda_n} v_n \leq -C\big(\mu_{\lambda^*} - \mu_n\big) qU - C\big(\lambda^* - \lambda_n\big)\eta U \leq 0 \text{ in } \Sigma_r. \tag{4.12}$$

mulplying the above inequality by v_n^+ and integrating by parts, we get that

$$\int_{\Sigma_r} |\nabla v_n^+|^2 dv_g - \mu_{\lambda_n} \int_{\Sigma_r} \rho^{-2} q |v_n^+|^2 dv_g + \lambda_n \int_{\Sigma_r} \eta \rho^{-2} |v_n^+|^2 dv_g \leq 0. \tag{4.13}$$

But Lemma 3.1 gives that

$$C \int_{\Sigma_r} \rho^{-2} \big(log \rho\big)^{-2} |v_n^+|^2 dv_g + \lambda_n \int_{\Sigma_r} \eta \rho^{-2} |v_n^+|^2 dv_g \leq 0. \tag{4.14}$$

Moreover $|\eta| < C\rho$ in Σ_r and $\lambda_n \searrow \lambda^*$ so bounded. Therefore there exists $r_0 > 0$ indenpend of n such that $v_n^+ \equiv 0$ in Σ_{r_0}. Thus we obtain (4.11). By the dominated convergence theorem, the fact that $u_n \longrightarrow u$ in $L^2(\mathcal{M})$ and (4.11) that

$$\rho^{-1} u_n \longrightarrow \rho^{-1} u \qquad \text{in } L^2(\mathcal{M}).$$

But

$$1 = \int_{\mathcal{M}} |\nabla u_n|^2 dv_g + o(1) = \mu_{\lambda_n} \int_{\mathcal{M}} \rho^{-2} q u_n^2 + \lambda_n \int_{\mathcal{M}} \rho^{-2} \eta u_n^2 dv_g + o(1), \tag{4.15}$$

taking the limit, we have

$$1 = \mu_{\lambda^*} \int_{\mathcal{M}} \rho^{-2} q u^2 + \lambda^* \int_{\mathcal{M}} \rho^{-2} \eta u^2 dv_g. \tag{4.16}$$

Hence $u \neq 0$ and it's a minimizer for μ_{λ^*}.

Proof of theorem 1.2

For the proof of this theorem the "if" part is given by Theorem 4.2 and the "only if" part is done in Theorem 4.1. □

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